PROPERTIES

OF

EXPANDING UNIVERSES

S.W.HAWKING

TRINITY HALL

plications and consequences of the expans
are examined. In Chapter 1 it is shown
on creates grave difficulties for the Hoy
ory of gravitation. Chapter 2 deals with
s of an expanding homogeneous and isotrop
ne conclusion is reached that galaxies ca
result of the growth of perturbations tha
all. The propagation and absorption of g
ation is also investigated in this appro
gravitational radiation in an expanding
by a method of asymptotic expansions. Th
r and the asymptotic group are derived.
e occurrence of singularities in cosmolo
s shown that a singularity is inevitable
very general conditions are satisfied.

nd even Einstein whose theory of relativi
r almost all modern developments in cosmo
ural to suggest a static model of the uni
e is a very grave difficulty associated w
such as Einstein's which is supposed to
n infinite time. For, if the stars had b
at their present rates for an infinite t
ave needed an infinite supply of energy.
radiation now would be infinite. Alterna
only a limited supply of energy, the whol
have reached thermal equilibrium which i
. This difficulty was noticed by Olbers w
to suggest any solution. The discovery
the nebulae by Hubble led to the abandon
s in favour of ones which were expanding.
there are several possibilities: the un
anded from a highly dense state a finite

s possible that the expansion may have be
t much the same rate for an infinite time
ry to postulate some form of continual cr
order to prevent the expansion from redu
ero. This leads to the 'steady-state' mo
anding presents the same appearance at al
ly cosmologies naturally placed man at or
f the universe, but, since the time of Co
demoted to a medium sized planet going r
star somewhere near the edge of a fairly
are now so humble that we would not claim
pecial position. However observations se
within experimental error (which is fai
e a spatially isotropic distribution arou
claiming any special position the distr
ropic about every point. This implies th
must be spatially homogeneous as well as
s homogeneity and isotropy hold only on

$$t^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin$$

$$k = 0 \text{ or } \pm 1$$

l be used extensively in the following ch
it will be shown that the Hoyle-Narlikar
n is incompatible with a metric of this
perturbations of this form will be consid
approximation and, in Chapter 3 gravitat
l be considered in a model which tends a
form.

of the Robertson-Walker models possess '
types: particle horizons and event hor
rizon is said to exist when an observer'
es not intersect the world line of every
se (or extended world line in the case o
h has been created). An example of a mo

n. Horizons will be further discussed in
eals with the occurrence of singularities
r connection with topology.

pter is self-contained and has its own r
notation is used throughout: space-tim
nnian manifold with metric tensor g_{ij}
ave signature -2 except in Chapter 2 whe
litate comparison with previous work, th
+2. Covariant differentiation is indic
Units are employed in which G, the spe
the gravitational constant, equal one.

wish to thank my supervisor, Dr. D. W. S
p and advice during my period of researc
would also like to thank Mr. B. Carter a
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R. G. McLenaghan for the calculation of
ntities in Chapter 3.
e research described in this thesis was
while I held a Research Studentship from

S. W. Hawking

r 1965 S. W. Hawking

ssertation is my original w

ion

cess of Maxwell's equations has led to
ics being normally formulated in terms o
grees of freedom independent of the part
er, Gauss suggested that an action-at-a-
ich the action travelled at a finite vel
sible. This idea was developed by Wheel
) who derived their theory from an actio
d only direct interactions between pairs
ature of this theory was that the 'pseud
re the half-retarded plus half-advanced
rom the world-lines of the particles. Ho
Feynman, and, in a different way, Hogart
show that, provided certain cosmologica
ere satisfied, these fields could combin
erved field. Hoyle and Narlikar (4) exte
neral space-times and obtained similar t

dary Condition

and Narlikar derive their theory from t

$$= \sum_{a \neq b}\sum \iint G(a,b)\, da\, db,$$

ntegration is over the world-lines of pa

In this expression G is a Green func

ies the wave equation:

$$')_{;ij}\, g^{ij} + \frac{1}{6} R\, G(X,X') = \frac{\delta^4(X,X'}{\sqrt{-g}}$$

the determinant of g_{ij} . Since the dou

on A is symmetrical between all pairs o

$,b$, only that part of $G(a,b)$ that i

between a and b will contribute to the

tion can be written

$$(X) m^{(b)}(X)] (R_{ik} - \tfrac{1}{2} g_{ik} R)$$

$$+ \sum_{a \neq b}\sum \tfrac{1}{3} [m^{(a)} (g_{ik} m^{(b)r}_{;r} - m^{(b)}_{;ik}) + 2(m$$

$$- \tfrac{1}{4} g_{ik} m^{(a);r} m^{(b)}_{;r}],$$

$) = \int G^*(x,a)\,da$. However, as a
of the particular choice of Green functi
of the field-equations is satisfied iden
us only 9 equations for the 10 component
em is indeterminate.

nd Narlikar therefore impose $\sum m^{(a)} = m_0$
equation. By then making the 'smooth-fl
n, that is by putting $\sum_{a \neq b}\sum m^{(a)} m^{(b)} \simeq$
the Einstein field-equations:

$$m_0^2 (R_{ik} - \tfrac{1}{2} R g_{ik}) = - T_{ik}$$

$$= \sum m^{(a)}(x) = \sum \int G^{*}(x, a)\, da$$

$$\sum \int G_{ret.}(x, a)\, da + \frac{1}{2} \sum \int G_{adv.}(x, a)$$

ment is highly restrictive; it will be
ot satisfied for the cosmological soluti
field-equations, and it appears that it
for any models of the universe that eit
nfinite amount of matter or undergo infi

ficulty is similar to that occurring in
eory when it is recognized that the univ
inite.

tonian potential ϕ obeys the equation:

$$\Box \phi = -\kappa \rho \qquad (\rho > 0),$$

...ng universe the retarded solution of the
finite by a sort of 'red-shift' effect. Th...
...ation will be infinite by a 'blue-shift'
...portant in Newtonian theory, since one is
...e solution of the equation and so may ign...
...anced solution and take simply the finite
...ation.

...y in the direct-particle interaction the...
...isfies the equation:

$$\Box m + \tfrac{1}{6} R m = N \qquad (N>0),$$

...he density of world-lines of particles.
... case, one may expect that the effect of
...the universe will be to make the retarde...
...e advanced solution infinite. However,
...to choose the finite retarded solution,
...erived from a direct-particle interactio...
...metric between pairs of particles, and o...

$$dt^2 - R^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \ldots\right.$$

e conformally flat, one can choose coord

y become

$$= \Omega^2\left[d\tau^2 - d\rho^2 + \rho^2\{d\theta^2 + \rho^2 \sin^2\right.$$

$$= \Omega^2 \eta_{ab}\, dx^a\, dx^b$$

is the flat-space metric tensor and

$$\rho) = \frac{R(t)}{\sqrt{\{[1+\frac{1}{4}K(\tau+\rho)^2][1+\frac{1}{4}K(\tau-}}$$

for the Einstein-de Sitter universe

$$= 0, \quad R(t) = \left(\frac{t}{T}\right)^{\frac{2}{3}} \quad (0 < t < \infty),$$

$$= R = \left(\frac{\tau}{T}\right)^2 \quad (0 < \tau < \infty),$$

nction $G^*(a, b)$ obeys the equation

$$G^*(a,b) + \frac{1}{6} R G^*(a,b) = \frac{\delta^4(a,b)}{\sqrt{-g}}$$

follows that

$$\left(\Omega^2 \eta^{ab} \frac{\partial}{\partial x^b} G^*\right) + \frac{\partial}{\partial x^a}\left(\eta^{ab} \frac{\partial}{\partial x^b} \Omega\right)$$

$$= \Omega^{-4} \delta^4(a,b)$$

$G^* = \Omega^{-1} S$, then

$$\frac{\partial}{\partial x^a}\left(\eta^{ab} \frac{\partial}{\partial x^b} S\right) = \delta^4(a,b)$$

ly the flat-space Green function equatio

$$(\tau_1, 0; \tau_2 \rho) = \frac{\Omega^{-1}(\tau_1)}{8\pi}\left[\frac{\delta(\rho - \tau_2 + \tau_1)}{\Omega(\tau_2)\rho} + \frac{\delta(\rho + \tau_2 - \tau_1)}{\Omega(\tau_2)\rho}\right],$$

is given by

$$= \int G^* N \sqrt{-g}\, dx^4 = \frac{1}{2}(m_{ret} + m$$

tegration is over the future light cone.

y be infinite in an expanding universe,

-de Sitter universe.

$$) = \left(\frac{\tau_1}{T}\right)^{-2} \int_{\tau_1}^{\infty} n(\tau_2 - \tau_1)\, d\tau_2$$

$$= \infty$$

y-state universe

$$(\tau_1) = \left(\frac{-T}{\tau_1}\right)^{-1} \int_{\tau_1}^{0} - n \left(\frac{T}{\tau_2}\right)^{3} ($$

$$= \infty$$

on the other hand, we have

steady-state universe

$$(\tau_1) = \left(\frac{-T}{\tau_1}\right)^{-1} \int_{-\infty}^{\tau_1} n \left(\frac{T}{\tau_2}\right)^3 (\tau_2 - \tau_1)$$

$$= \frac{1}{2} n T^2$$

be seen that the solution $m =$ const.

$$\Box m + \frac{1}{6} R m = N$$

cosmological metric, the half-advanced

solution since this would be infinite.

of the Einstein-de Sitter and steady-stat

e retarded solution.

ield

d Narlikar derive their direct-particle

theory of the 'C'-field from the action

$$(X, X') = \frac{\delta^4(X, X')}{\sqrt{-g}}.$$

e 'C'-field by

$$C(x) = \sum \int \hat{G}(x,a)_{;i_a} \, da^i,$$

er-current J^k by

$$J^k(y) = \sum \int \delta^4(y, b) \, db^k.$$

$$C(x) = \int \hat{G}(x,y) J^k(y)_{;k} \sqrt{-g}$$

$$\Box C = J^k_{\ ;k}$$

that the sources of the 'C'-field are th

is created or destroyed.

he case of the 'm'-field, the Green fun

-symmetric, that is

$$\hat{G}(a,b) = \tfrac{1}{2}\hat{G}_{ret}(a,b) + \tfrac{1}{2}\hat{G}_{a}$$

se on a large scale although there may b
es. In this universe, the value of C wi
ts gradient time-like and of unit magnit
nis universe, we may check it for consis
the advanced and retarded 'C'-fields and
is finite. We shall not do this direct
at the advanced field is infinite while
finite.

r a region in space-time bounded by a th
space-like hypersurface D at the presen
light cone Σ of some point P to the

s's theorem

$$\square C \sqrt{-g}\, dx^4 = \int_{\Sigma + D} \frac{\partial C}{\partial n} dS$$

$$= \int J^{\kappa}{}_{;\kappa} \sqrt{-g}\, dx^4$$

s the rate of creation of matter= n (cor

tate universe, and hence

$$\int_{D} \frac{\partial C'}{\partial n} dS = nV.$$

P is taken further into the future, th
n V tends to infinity. However, the are
D tends to a finite limit owing to hor
erefore the gradient $\frac{\partial C'}{\partial n}$ must be
lculation shows the gradient of the reta
finite. Their sums cannot therefore giv
t gradient required by the Hoyle-Narlika
orth noting that this result was obtaine
mptions of a smooth distribution of matt
lflatness.

boundary conditions for them. Thus it d
e model for the universe but allows a wh
Clearly a theory that provided boundary
tricted the possible solutions would be
The Hoyle-Narlikar theory does just tha
that $m = \frac{1}{2} m_{ret} + \frac{1}{2} m_{adv}$ is
o a boundary condition). Unfortunately,
ove, this condition excludes those model
espond to the actual universe, namely th
lker models.

culations given above have considered th
led with a uniform distribution of matte
e if we are able to make the 'smooth-flu
n to obtain the Einstein equations. Alt
oximation is invalid, it cannot be said
s the Einstein equations.

t possibly be that local irregularities

field will ...

.

on that it is possible to formulate a di

raction theory of electrodynamics that d

s difficulty of having the advanced solu

hat in electrodynamics there are equal n

positive and negative sign. Their fiel

ther out and the total field can be zero

regularities. This suggest that a possi

oyle-Narlikar theory would be to allow n

and negative sign. The action would be

$$\sum_{a \neq b} \sum q_a q_b \iint G^*(a, b) \, da \, db \qquad (q_a$$

b are gravitational charges analogo

ges. Particles of positive q in a pos

nd particles of negative q in a negati positive q . Its own gravitational

be to repel all other particles. Thus

erties of the negative mass described by

tive gravitational mass and negative ine

ere does not seem to be any matter havin

ies in our region of space (where $m \triangleq$

early be separation on a very large scal

possible to identify particles of negati

er, since it is known that antimatter ha

. Hwever, the introduction of negative

y raise more difficulties than it would s

r and ynman	Rev.Mod.Phys. 21 425
n	Proc.Roy.Soc.A 267 36
l ikar	Proc.Roy.Soc.A 277 1
l ikar	Proc.Roy.Soc.A 282 17
l ikar	Proc.Roy.Soc.A 282 19
nd	Phys.Rev. 68 250 1945
	Rev.Mod.Phys.29 423 1

-atively by Lifshitz[2], Liftshitz a
nd Irvine[4]. Their method was to conside
e metric tensor. This has the disadvantag
not a physically significant quantity, si
neasure it, but only its second derivative
what the physical interpretation of a giv
the metric is. Indeed it need have no phy
ll, but merely correspond to a coordinate
ad it seems preferable to deal in terms o
the physically significant quantity, the

s represented as a four-dimensional Riema
r g_{ab} of signature +2. Covariant diffe
indicated by a semi-colon. Square bracke
antisymmetrisation and round brackets sym
or the Riemann and Ricci tensors are:-

e energy momentum tensor of matter. We w

consists of a perfect fluid. Then,

$$= \mu U_a U_b + p h_{ab}$$

velocity of the fluid, $U_a U^a = -1$:

μ is the density

p is the pressure

$h_{ab} = g_{ab} + U_a U_b$ is the projection oper

ane orthogonal to U_a :

$$h_{ab} U^b = 0.$$

gradient of the velocity vector U_a as

$$\omega_{ab} + \sigma_{ab} + \tfrac{1}{3} h_{ab} \theta - \dot{U}_a U_b$$

$U_{a;b} U^b$ is the acceleration

$U_a{}^{;a}$ is the expansion,

$U_{(c;d)} h^c_a h^d_b - \frac{1}{3} h_{ab} \theta$ is the shear

$U_{[c;d]} h^c_a h^d_b$ is the rotat

We define the rotation vector ω as

$C_{\circ bpq}$ ...

$\frac{1}{2}\, C_a{}^{pqr}\, \eta_{qrbs}\, u_p\, u^s \quad ,$

$8\, u_{[a}\, E_{b]}{}^{[c}\, u^{d]} - 4\, \delta^{[c}_{[a}\, E^{d]}_{b]} \quad ,$

$- 2\, \eta_{abcd}\, u^p\, H^{q[c}\, u^{d]} - 2\, \eta^{cdrs}\, u_r\, H_{s[a}\, u_{b]}$

$_{(ab)} \quad , \qquad H_{ab} = H_{(ab)} \quad ,$

$H_a{}^a = 0$

$H_{ab}\, u^b = 0 \quad ,$

each have five independent components.

ianchi identities,

$$3\, E_{ab}\,\omega^{b} - \eta_{abcd}\, u^{b}\, \sigma^{c}{}_{e}\, E \quad = (\mu + p)\, \omega_{a}$$

$$_{b)cde}\, u^{c}\, H_{f}{}^{d;e} + E_{ab}\,\theta - E^{c}{}_{(a}\,\omega_{b)c}$$

$$_{b)c} - \eta_{acde}\,\eta_{bpqr}\, u^{c}\, u^{p}\, \sigma^{dq}\, E^{er}$$

$$\eta_{bcde}\, u^{c}\, \dot{u}^{e} = -\tfrac{1}{2}(\mu + p)\, \sigma_{ab} \qquad ,$$

$$_{b)cde}\, u^{c}\, E_{f}{}^{d;e} + H_{ab}\,\theta - H^{c}{}_{(a}\,\omega_{b)c}$$

$$_{b)c} - \eta_{acde}\,\eta_{bpqr}\, u^{c}\, u^{p}\, \sigma^{dq}\, H^{er}$$

$$\eta_{bcde}\, u^{c}\, \dot{u}^{e} = 0 \qquad .$$

ates projection by h_{ab} orthogonal to U
7)).

cted Bianchi identities give,

$$(R_{ab} - \tfrac{1}{2}\, g_{ab}\, R)^{;b} = - T_{ab}{}^{;b} = 0 \qquad ,$$

$$\dot{\mu} + (\mu + p)\,\theta = 0 \qquad ,$$

$$(\mu + p)\,\dot{u}_{a} + p_{;b}\, h^{b}{}_{a} = 0 \qquad .$$

$${}_b - \omega_{ac}\omega^c{}_b - \sigma_{ac}\sigma^c{}_b - \tfrac{2}{3}\sigma_{ab}\theta$$

$$h_{ab}(2\omega^2 - 2\sigma^2 + \dot{u}_c{}^{;c}) + \dot{u}_a\dot{u}_b$$

$${}_{(p;q)} h^p_a h^q_b ,$$

$$\omega_{ab}\omega^{ab} , \quad 2\sigma^2 = \sigma_{ab}\sigma^{ab}$$

what may be regarded as equations of cons

$$\left(\omega^b{}_{c;b} + \sigma^b{}_{c;b} \right) h^c_a - \dot{u}^b \left(\omega_{ab} + \sigma_{ab} \right) \Big]$$

$${}_a \dot{u}^a ,$$

$$\eta_{b)cde} u^c \left[\omega_f{}^{d;e} + \sigma_f{}^{d;e} \right] .$$

turbations of a universe that in the undi

flat, that is

$$C_{abcd} = 0 .$$

iverse we consider small perturbations o

nd of the Weyl tensor. We neglect product

perform derivatives with respect to the u

all the quantities we are interested in v

ne scalars, μ, p, θ have unperturbed value

tions that merely represent coordinate tra

ysical significance.

rst order the equations (1) - (4) and (7)

$${}_{a}{}^{b}\mu_{;b} \quad ,$$

$$2)\,\omega_a \quad ,$$

$$+ h^f{}_{(a}\eta_{b)cde}u^c H_f{}^{d;e} = -\tfrac{1}{2}(\mu+p)\,\sigma_{ab}$$

$$- h^f{}_{(a}\eta_{b)cde}u^c E_f{}^{d;e} = 0 \quad ,$$

$${}^2 + \dot{u}_a{}^{;a} - \tfrac{1}{2}(\mu+3p) \quad ,$$

$$\omega_{ab}\theta + \dot{u}_{[p;q]}h^p_a h^q_b \quad ,$$

he unperturbed state the rotation and acc
ust be hypersurface orthogonal.

$$u_a = \tau_{;a} \quad ;$$

ures the proper time along the world lines.
constant are homogeneous and isotropic the
constant curvature. Therefore the metric

$$ds^2 = -d\tau^2 + \Omega^2 d\gamma^2$$

$$\Omega = \Omega(\tau) \quad ,$$

$d\gamma^2$ is the line element of
zero or unit positive or nega

by,

$$\frac{dt}{d\tau} = \frac{1}{\Omega} \quad ,$$

$$ds^2 = \Omega^2(-dt^2 + d\gamma^2) \quad .$$

$$u_a = (-\Omega, 0, 0, 0) \quad ;$$

relation between μ and p, we may deter

er the two extreme cases, $p = 0$ (dust) an

Any physical situation should lie between

$$\frac{M}{\Omega^3} \qquad M = \text{const.}$$

$$\frac{\ddot{\Omega}}{\Omega} - \frac{1}{2\Omega^3} = 0 ,$$

$$\Omega^2 - \frac{1}{\Omega} = E \quad , \qquad E = \text{const.}$$

E 0,

$$\cosh\sqrt{\frac{EM}{3}}\,t - 1\Big) , \qquad \tau = \frac{1}{2E}\Big(\sqrt{\frac{3}{EM}}\ \sinh$$

E = 0,

$$, \qquad \tau = \frac{M}{36}\,t^3 ;$$

E 0,

$$- \cos\sqrt{\frac{-EM}{3}}\,t\Big) , \qquad \tau = \frac{-1}{2E}\Big(t - \sqrt{\frac{3}{-EM}}\ s$$

$^*R = -\frac{}{\Omega^2}$, $M = \frac{}{E}$;

$^*R = 0$;

$^*R = \frac{6}{\Omega^2}$, $M = \frac{-3}{E}$.

$\dot{\mu} = -4\frac{\dot{\Omega}}{\Omega}$,

$\frac{3\ddot{\Omega}}{\Omega} = -\mu$,

$\mu = \frac{M}{\Omega^4}$, $\therefore \frac{3\dot{\Omega}^2}{M} - \frac{1}{\Omega^2} = E$

> 0,

t , $\tau = \frac{1}{E}(\cosh t - 1)$, $^*R = -$

$= 0$,

$\tau = \frac{1}{2}t^2$, $^*R = 0$

0,

t , $\tau = \frac{1}{E}(\cos t - 1)$, $^*R =$

$$\dot{\omega} = -\omega\left(\tfrac{2}{3}\theta + \tfrac{1}{4}\tfrac{\dot{\mu}}{\mu}\right),$$

$$= -\tfrac{1}{3}\omega\theta ,$$

$$\omega = \frac{\omega_0}{\Omega}$$

dies away as the universe expands. This i

he conservation of angular momentum in an

ons of Density

) we have the equations,

$$\dot{\mu} = -\mu\theta$$

$$\dot{\theta} = -\tfrac{1}{3}\theta^2 - \tfrac{1}{2}\mu$$

no spatial derivatives. Thus the behaviou

fected by the behaviour of another. Pertu

n some regions having slightly higher or l e time at which the whole universe begins

any real instability when E = 0. This ca

elative to all the possible values E can

annot really be used as an arguement to d

be some reason why the universe should h

ith energy $-\delta E$, in a universe with E =

$$= \frac{1}{4\delta E}\left(t^2 - \frac{t^4}{12} + \cdots\right)$$

$$= \frac{1}{12\delta E}\left(t^3 - \frac{t^5}{20} + \cdots\right)$$

$$= \frac{3}{\delta E\,\Omega^3} = \frac{4}{3}\tau^{-2}\left(1 + \frac{(\delta E)^{2/3}}{2\sqrt{3}}\tau^{2/3} + \cdots\right)$$

$$= \frac{4}{3}\tau^{-2}$$

rbation grows only as $\tau^{2/3}$. This is not

axies from statistical fluctuations even

However, since an evolutionary universe h

er[8], Penrose[9]) different parts do not turbation cannot contract unless it has a

action of the pressure forces make it st

to contract. Eliminating θ,

$$\ldots^2 - \frac{4}{3}\mu^3 + \frac{4}{3}\mu^2 \dot{u}_a{}^{;a} = 0$$

$$\ldots_{a;b} h^{ab} + \dot{u}_a \dot{u}^a$$

$$\frac{1}{4} \frac{h^{ac}(h^b{}_a \mu_{;b})_{;c}}{\mu}$$

to our

is the Laplacian in the hypersurface τ

perturbation as a sum of eigenfunctions

here,

$$S^{(n)}{}_{;c} u^c = 0$$

$$h^{ac}(h^b{}_a S^{(n)}{}_{;b})_{;c} = -\frac{n^2}{\Omega^2} S^{(n)}$$

ions will be hyperspherical and pseudohype

$$-\frac{1}{2} B \quad \mu_o - B \quad [\frac{1}{3}\mu_o - \frac{}{3\Omega^2}\mu_o] -$$

$\frac{n^2}{4\Omega^2}$, $B^{(n)}$ will grow.

2

$$B^{(n)} \doteq C\tau + D\tau^{-1}$$

ons grow for as long as light has not had
cant distance compared to the scale of th
that time pressure forces cannot act to

$$\gg \mu_o \; , \qquad B''^{(n)} + B'^{(n)}\frac{\Omega'}{\Omega} + \frac{n^2}{3}B^{(n)}$$

$$) \doteq C\,\Omega^{-\frac{1}{2}} e^{i\frac{n}{\sqrt{3}}t}$$

waves whose amplitude decreases with time
those obtained by Lifshitz and Khalatniko

tate Universe

he steady-state universe we must add extr tum tensor. Hoyle and Narlikar[10] use,

$$u_a u_b + p h_{ab} - C_a C_b + \tfrac{1}{2} g_{ab} C_d C^d .$$

$$C_a = C_{;a} \quad ,$$

$$C_{;a}{}^{a} = - j_a{}^{;a} \quad ,$$

$$j_a = (\mu + p)\, u_a$$

,

$$\dot{\mu} + (\mu + p)\,\theta + u^a C_a C_b{}^{;b} = 0$$

$$(\mu + p)\,\dot{u}_a + p_{;b} h^b{}_a - h^b{}_a C_b C_d{}^{;d} = 0$$

is a scalar, this implies that the rotati

On the other hand if (23) does not hold,

e (c.f. Raychaudhuri and Bannerjee[11]).

te set of equations we will adopt (23) bu

C_a is the gradient of a scalar. The c

sfactory but it is difficult to think of

oyle and Narlikar[12] seek to avoid this

icle rather than a fluid picture. Howeve

since it leads to infinite fields (Hawkin

$$= - u_a \left[1 - \frac{\dot{p}}{\dot{\mu} + \dot{p} + (\mu + p)\theta} \right]$$

$$- (\dot{\mu} + \dot{p}) - (\mu + p)\theta \qquad ,$$

$$- \theta \left[1 - \frac{\dot{p}}{\dot{\mu} + \dot{p} + (\mu + p)\theta} \right] + \left(\frac{\dot{p}}{\dot{\mu} + \dot{p} + (\mu +}\right.$$

therefore rotational perturbations also

w becomes

$$\dot{\theta} = -\frac{1}{3}\theta^2 - \frac{1}{2}(\mu + 3p) + 1$$

$$\theta \rightarrow \sqrt{3(\tfrac{1}{2} - p)}$$

nfirm those obtained by Hoyle and Narlika

at galaxies cannot be formed in the stead

growth of small perturbations. However t

ibility that there might by a self-perpet

perturbations which could produce galaxi

burgh and Saffman(16)).

l Waves

ider perturbations of the Weyl tensor tha

ional or density perturbations, that is,

$$^{b} = H_{ab}{}^{;b} = 0$$

ith a non-expanding congruence U^a this r
f the linearised theory,

$$\Box^2 E_{ab} = 0$$

term in (24) is the Laplacian in the hyper
acting on E_{ab}. We will write E_{ab} as a
of this operator.

$$E_{ab} = \sum A^{(n)} V_{ab}{}^{(n)}$$

$$= 0 \quad ,$$

$$h^c_f h^d_g h^e_k)_{;i} h^{ki} h^f_a h^g_b = \frac{-n^2}{\Omega^2} V_{ab}{}^{(n)}$$

$$= 0 \quad , \qquad V_a{}^a = 0 \quad .$$

$$\sum \frac{A'^{(n)}}{\Omega} V_{ab}^{(n)}$$

(24)

$$A'^{(n)} + A^{(n)} \left[n^2 + 3\frac{\Omega''}{\Omega} + 6\frac{\Omega'^2}{\Omega^2} + \frac{1}{3}(\mu +\right.$$

$$\left[\Omega'(\mu + p) + \frac{1}{2}\Omega^2(\dot{\mu} + \dot{p}) \right] = 0$$

iate again and substitute for D',

$$\Omega \gg \frac{1}{n^2}$$

$$\simeq \frac{1}{\Omega^3} e^{int}$$

onal field E_{ab} decreases as Ω^{-1} and t

H^{ab}) as Ω^{-6}. We might expect this as

es may be written, to the linear approxima

$$\frac{\partial}{\partial x^e}\left(\Omega^{-1} C_{abcd} \right) = J_{abc} .$$

interaction with the matter could be negl

proportional to Ω and E_{ab}, H_{ab} to Ω

dy-state universe when μ and θ have n as measured by the energy momentum pseudo

g Cartesian coordinate system which depend

s. Since the frequency will be inversely

rgy measured by the pseudo-tensor will be

other rest mass zero fields.

f Gravitational Waves

seen, gravitational waves are not absorbed

Suppose however there is a small amount of

this by the addition of a term $\lambda \sigma_{ab}$

tensor, where λ is the coefficient of vi

$$T_{ab}{}^{;b} = 0$$

$$(\mu + p)\theta - 2\lambda\sigma^2 = 0$$

$$h^{f}{}_{(a}\eta_{b)cde}\, u^{c}\, E_{f}{}^{d;e} = -\tfrac{1}{2}\lambda H_{ab}.$$

on the right of equations (27), (28) are

in Maxwell's equations and will cause th

ctor $e^{-\frac{\lambda}{2}t}$. Neglecting expansion for

a wave of the form,

$$_{ab} = {}_{0}E_{ab}\, e^{i\nu\tau}.$$

orbed in a characteristic time $2/\lambda$ inde

25) the rate of gain of rest mass energy

$2\lambda\sigma^{2}$ which by (19) will be $2\lambda\, {}_{0}E^{2}\nu^{-2}$

in the wave is $4\, {}_{0}E^{2}\nu^{-2}$. This confi

able energy of gravitational radiation wi

expanding universe. From this we see tha

diation behaves in much the same way as o

. In the early stages of an evolutionary

avitational radiation does not contribute
momentum tensor T_{ab}. Nevertheless it wi
onal effect. By the expansion equation,

$$\frac{1}{3}\theta^2 - 2\sigma^2 - \frac{1}{2}(\mu + 3p) .$$

ravitational radiation at frequency ν,

$$_{\circ}E^2 \nu^{-2}$$

ensity of the radiation is $4\ _{\circ}E^2 \nu^{-2}$

$$\frac{1}{3}\theta^2 - \frac{1}{2}\mu_G - \frac{1}{2}(\mu + 3p)$$

e gravitational "energy" density. Thus gr
active attractive gravitational effect.
this seems to be just half that of electr

suggested by Hogarth[18] and Hoyle and Na
e a connection between the absorption of r
Time. Thus in universes like the steady-

so for the steady-state universe since λ

olutionary universes λ will be a functio

omplete absorption if $\int \lambda d\tau$ diverges. No

is the temperature. For a monatomic gas

tegral will diverge (just). However the e

ty assumed that the mean free path of the

o the scale of the disturbance. Since the

and the wavelength $\propto \Omega^{-1}$, the mean fr

eater than the wavelength and so the effec

re rapidly than Ω^{-1}. Thus there will n

he theory would not predict retarded solut

slightly academic since gravitational radi

l, let alone investigated to see whether i

advanced solution.

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ational radiation in empty asymptoticall
een examined by means of asymptotic expa
of authors.[1-4] They find that the di
of the outgoing radiation field "peel of
as different powers of the affine radia
es to investigate how this behaviour is
ence of matter, one is faced with a diff
ot arise in the case of, say, electromag
n matter. For this one can consider the
through an infinite uniform medium that
the disturbance created by the radiation
vitational radiation this is not possibl
am were initially static, its own self g
it to contract in on itself and it would
Hence one is forced to investigate gravi
matter that is either contracting or ex
Chapter 2, we identify the Weyl or confo

zero.

id essentially non-gravitational phenome

ves, we will consider gravitationalradia

through dust. It was shown in Chapter 2

flat universe filled with dust must hav

$$ds^2 = \Omega^2(dt^2 - d\rho^2 - \sin^2\rho(d\theta^2 + \sin$$

$$\Omega = A(1 - \cos t)$$

$$ds^2 = \Omega^2(dt^2 - d\rho^2 - \rho^2(d\theta^2 + \sin$$

$$\Omega = \tfrac{1}{2}At^2 \quad (1$$

$$ds^2 = \Omega^2(dt^2 - d\rho^2 - \sinh^2\rho(d$$

$$\Omega = A(\cosh t - 1) \quad (1.$$

represents a universe in which the mat

the initial singularity with insuffici

al case. D. Norman (5) has investigated
" behaviour in this case using Penrose's
). He was however forced to make certai
he movement of the matter which will be
oreover, he was misled by the special na
-De Sitter universe in which affine and
ffer. Another reason for not considerin
ein-De Sitter universe is that it is uns
of a gravitational wave will cause it to
ally and develop a singularity.

therefore consider radiation in a unive
ch corresponds to the general case where
panding with more than enough energy to
again.

an-Penrose Formalism

y the notation of Newman and Penrose.(3
$l^{\mu}\ n^{\mu}\ m^{\mu}\ \bar{m}^{\mu}$

se vectors with a tetrad index

$^{\mu}, n^{\mu}, m^{\mu}, \bar{m}^{\mu})$ $\qquad a = 1, 2, 3, 4.$

es are raised and lowered with the metri

$$\eta^{ab} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad (2.1$$

$$^{\mu\nu} = \eta^{ab} \underset{a}{Z}^{\mu} \underset{b}{Z}^{\nu}$$

$$l^{\mu} n^{\nu} + n^{\mu} l^{\nu} - m^{\mu} \bar{m}^{\nu} - \bar{m}^{\mu} m^{\nu}$$

on coefficients are defined by:

$$= \underset{a}{Z}_{\mu;\nu} \overset{b}{Z}^{\mu} \overset{c}{Z}^{\nu} \qquad (2$$

bac

$$\gamma_{131} = L_{\mu;}$$

$$-\gamma_{241} = -n_{\mu;\nu}\bar{m}^{\mu}L^{\nu}$$

$$\frac{1}{2}\left(\gamma_{121} - \gamma_{341}\right) = \frac{1}{2}\left(L_{\mu;\nu}n^{\mu}L\right.$$

$$\gamma_{134} = L_{\mu;\nu}m^{\mu}\bar{m}^{\nu}$$

$$-\gamma_{244} = n_{\mu;\nu}\bar{m}^{\mu}\bar{m}^{\nu}$$

$$\frac{1}{2}\left(\gamma_{124} - \gamma_{344}\right) = \frac{1}{2}\left(L_{\mu;\nu}n^{\mu}\bar{m}^{\nu} - m\right.$$

$$\frac{1}{2}\left(\gamma_{123} - \gamma_{343}\right) = \frac{1}{2}\left(L_{\mu;\nu}n^{\mu}m^{\nu} - m_{\mu}\right.$$

$$\gamma_{133} = L_{\mu;\nu}m^{\mu}m^{\nu}$$

$$-\gamma_{243} = -n_{\mu;\nu}\bar{m}^{\mu}m^{\nu}$$

$$-\gamma_{242} = -n_{\mu;\nu}\bar{m}^{\mu}n^{\nu}$$

$_{\mu} = u_{;\mu}$. Thus L_μ wi

irrotational. This implies

$$\kappa = 0$$
$$\rho = \bar{\rho}$$
$$\epsilon = -\bar{\epsilon} \qquad (3.$$
$$\tau = \bar{\alpha} + \beta$$

$m^\mu, \bar{m}^\mu$ to be parallelly tr

. This gives

$$\pi = \epsilon = 0 \qquad (3.3)$$

coordinate we take an affine parameter

desics L_μ

$$r_{;\mu} L^\mu = 1 \qquad (3$$

^{4} are two coordinates that label the g

ce u = const.

$$X^3_{;\mu} L^\mu = X^4_{;\mu} L^\mu = 0 \qquad ($$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & g^{22} & g^{23} & g^{24} \\ 0 & g^{23} & g^{33} & g^{34} \end{bmatrix} \qquad (3.6)$$

$$- \gamma_{abd;c} + \gamma_{e} \gamma - \gamma_{e} \gamma + \gamma (\gamma_{e} - \gamma_{e}$$

$$C_{bcd} + \eta_{a[d} R_{c]b} + \eta_{b[c} R_{d]a} + \frac{R}{3} \eta_{a[}$$

mbinations of rotation coefficients alre

with $\kappa = \pi = \epsilon = 0$ we have

$$^{2} + \sigma\bar{\sigma} + \phi_{00} \qquad (3.$$

$$\rho\sigma + \Psi_0 \qquad (3$$

$$\rho + \bar{\tau}\sigma + \Psi_1 + \Phi_{01} \qquad (3$$

$$\rho + \beta\bar{\sigma} + \Phi_{10} \qquad (3$$

$$\rho + \alpha\sigma + \Psi_1 \qquad (3$$

$$\alpha + \bar{\tau}\beta + \Psi_2 - \Lambda + \Phi_{11} \qquad (3.$$

$$\rho + \mu\bar{\sigma} + \Phi_{20} \qquad (3.$$

$$\mu\rho + \lambda\sigma + \Psi_2 + 2\Lambda \qquad (3.$$

$$\tau\lambda + \bar{\tau}\mu + \Psi_3 + \Phi_{21} \qquad (3.1$$

$$\rho - \lambda\sigma - 2\alpha\beta + \alpha\bar{\alpha} + \beta\bar{\beta} - \Psi_2 + \Lambda +$$

$$\alpha + \bar{\beta})\mu + (\bar{\alpha} - 3\beta)\lambda - \Psi_3 + \Phi_{21}$$

$$\gamma\mu - 2\nu\beta + \bar{\gamma}\mu + \mu^2 + \lambda\bar{\lambda} + \Phi_{22}$$

$$\tau\mu - \sigma\nu + (\mu - \gamma + \bar{\gamma})\beta + \bar{\lambda}\alpha$$

$$2\tau\beta + (\bar{\gamma} + \mu - 3\gamma)\sigma + \bar{\lambda}\rho + \Phi_{02}$$

$$(\gamma + \bar{\gamma} - \bar{\mu})\rho - 2\alpha\tau - \lambda\sigma - \Psi_2$$

$$\rho\nu - \tau\lambda - \lambda\beta$$
$$+ (\bar{\gamma} - \gamma - \bar{\mu})\alpha - \Psi_3$$

$$= M^{\mu} \nabla_{\mu} = \omega \frac{\partial}{\partial r} + \xi^{i} \frac{\partial}{\partial x^{i}}$$

$$= -\frac{1}{2} R_{11} = \bar{\phi}_{00}$$

$$= -\frac{1}{4} \left(R_{12} + R_{34} \right) = \bar{\phi}_{11}$$

$$= -\frac{1}{2} R_{13} = \bar{\phi}_{10}$$

$$= -\frac{1}{2} R_{23} = \bar{\phi}_{21}$$

$$= -\frac{1}{2} R_{33} = \bar{\phi}_{20}$$

$$= -\frac{1}{2} R_{22} = \bar{\phi}_{22}$$ (3

$$= \frac{R}{24}$$ (3.

$$-\underset{1213}{C} = -C_{\alpha\beta\gamma\delta}\, l^{\alpha} n^{\beta} l^{\gamma} m^{\delta}$$

$$-\frac{1}{2}\left(\underset{1212}{C} + \underset{1234}{C}\right) = -\frac{1}{2} C_{\alpha\beta\gamma\delta}$$

$$\times\left(l^{\alpha} n^{\beta} l^{\gamma} n^{\delta} + l^{\alpha} n^{\beta} m^{\gamma} \bar{m}\right.$$

$$\underset{224}{C} = C_{\alpha\beta\gamma\delta}\, l^{\alpha} n^{\beta} n^{\gamma} \bar{m}^{\delta}$$

$$-\underset{2424}{C} = -C_{\alpha\beta\gamma\delta}\, n^{\alpha} \bar{m}^{\beta} n^{\gamma} \bar{m}$$

$$= \rho\omega + \sigma\bar{\omega} - (\bar{\alpha} + \beta) \qquad (3.$$

$$\tau\bar{\xi}^i + \bar{\tau}\xi^i \qquad (3.$$

$$\tau\bar{\omega} + \bar{\tau}\omega - (\gamma + \bar{\gamma}) \qquad (3.$$

$$\xi^i = (\mu + \bar{\gamma} - \gamma)\xi^i + \bar{\lambda}\bar{\xi}^i \qquad (3$$

$$\bar{\xi}^i = (\bar{\beta} - \alpha)\xi^i + (\bar{\alpha} - \beta)\bar{\xi}^i \qquad ($$

$$\omega = (\bar{\beta} - \alpha)\omega + (\bar{\alpha} - \beta)\bar{\omega} + (\mu -$$

$$\omega = (\mu + \bar{\gamma} - \gamma)\omega + \bar{\lambda}\bar{\omega} - \bar{\nu}$$

$$D\Psi_1 + D\Phi_{01} - \delta\Phi_{00} = 4\alpha\Psi_0 - 4\rho\Psi_1 - ($$

$$+ 2\rho\Phi_{01} + 2\sigma\Phi_{10}$$

$$D\Phi_{02} - \delta\Phi_{01} = (4\gamma - \mu)\Psi_0 - 2(2\tau +$$

$$3\sigma\Psi_2 - \bar{\lambda}\Phi_{00} - 2\beta\Phi_{01} + 2\sigma\Phi_{11} + ($$

$$\Psi_3) + 2(D\Phi_{11} - \delta\Phi_{10}) + \bar{\delta}\Phi_{01} - \Delta\Phi_{00} = 3$$

$$\Psi_1 + (\bar{\mu} - 2\mu - 2\gamma - 2\bar{\gamma})\Phi_{00} + (2\alpha$$

$$\tau - 2\bar{\alpha})\Phi_{10} + 2\rho\Phi_{11} + 2\sigma\Phi_{20} - \bar{\sigma}\Phi$$

$$\Psi_2) + 2(D\Phi_{12} - \delta\Phi_{11}) + (\bar{\delta}\Phi_{02} - \Delta\Phi_{01})$$

$$+ 6(\gamma - \mu)\Psi_1 - 9\tau\Psi_2 + 6\sigma\Psi_3$$

$$2(\bar{\mu} - \mu - \gamma)\Phi_{01} - 2\lambda\Phi_{10} + 2\tau$$

$$(2\alpha + \bar{\tau} - 2\bar{\beta})\Phi_{02} + 2\sigma\Phi_{21}$$

$$\Psi_3) + (D\Phi_{21} - \delta\Phi_{20}) + 2(\bar{\delta}\Phi_{11} - \Delta\Phi$$

$$6\rho\Psi_3 - 2\nu\Phi_{00} + 2\lambda\Phi_{01} + 2(\bar{\mu} -$$

$$+\bar{\delta}\,\phi_{21} - \Delta\phi_{20} = 3\lambda\Psi_2 - 2\alpha\Psi$$
$$\nu\,\phi_{01} + 2\lambda\,\phi_{11} + (2\gamma - 2\bar{\gamma} +$$
$$(\bar{\tau} - \alpha)\,\phi_{21} - \bar{\sigma}\,\phi_{22} \qquad (\underline{3.57})$$

$$+\bar{\delta}\,\phi_{22} - \Delta\,\phi_{21} = 3\nu\Psi_2 - 2($$
$$(4\beta - \tau)\,\Psi_4 - 2\nu\,\phi_{11} - \bar{\nu}\,\phi_{20}$$
$$2(\gamma + \bar{\mu})\,\phi_{21} + (\bar{\tau} - 2\bar{\beta} - 2$$

$$\phi_{02} + \Delta\phi_{01} + 3\delta\Lambda = (2\gamma - \mu - 2\bar{\mu})$$
$$\phi_{11} + (2\bar{\beta} - 2\alpha - \bar{\tau})\,\phi_{02} + 3\rho\,\phi_{12} + \sigma\,\phi$$

$$\phi_{01} + \Delta\,\phi_{00} + 3D\Lambda = (2\gamma - \mu + 2\bar{\gamma} - \bar{\mu})$$
$$\tau)\,\phi_{10} + 4\rho\,\phi_{11} + \bar{\sigma}\,\phi_{02} + \sigma\,\phi_{20}$$

$$-\bar{\delta}\,\phi_{12} + \Delta\,\phi_{11} + 3\Delta\Lambda = \nu\,\phi_{01} + \bar{\nu}\,\phi$$

$$\Omega = A(\cosh t - 1)$$

$-\rho$

$$\Omega^2 \left[-du^2 + 2du\,dt - \sinh^2(t-u \right.$$

l coordinate

late r, the affine parameter, we note t

parameter for the metric within the squa

erefore $\quad r = \int \Omega^2 dt + B(u, \theta, \phi$

fine parameter for $(4\cdot 1)$

nstant along the null geodesic. Normall

n so that $r = 0$ when $t = u$. Howe

t will be more convenient to make it zer

s

$$r = \int_0^t \Omega^2 dt'$$

at surfaces of constant r are surface

. This may seem rather odd, but it sho

t that the choice of B will not affec

endence of quantities. That is, if

$$q = A^2\left[\frac{1}{4}\sinh 2t - 2\sinh t + \frac{3}{2}\right.$$

in the universe is assumed to be dust so

r may be written

$$T_{ab} = \mu V_a V_b$$

sturbed case, from Chapter 2

$$\mu = \frac{6A}{\Omega^3}$$

$$V_a = \Omega t_{;a} \qquad V_a V^a =$$

$$= \sqrt{2}\, s + A - \frac{3A^2}{2\sqrt{2}} \frac{\log s}{s} + O(s^{-1}$$

$$= q$$

we try to expand μ as a series in pow

ill be very messy and will involve terms

$$\frac{\log^m s}{s^n} \quad *$$

---

e pointed out that the expansions used w

ned to be valid asymptotically. They wi

undisturbed case. That is

$$= A(\cosh t - 1)$$

$$A^2\left[\frac{1}{4}\sinh 2t - 2\sinh t + \frac{3}{2}t\right]$$

$$= \frac{\sqrt{1 + \frac{2A}{\Omega}}}{\Omega}$$

$$= \frac{1}{\Omega}\left[1 + \frac{A}{\Omega} - \frac{A^2}{2\Omega^2} + \frac{A^3}{2\Omega^3} - \frac{5A^4}{8\Omega^4}\right.$$

and fourth coordinates it is more conve

phic coordinates than spherical polars.

e matter is dust its energy-momentum ten

Ricci-tensor have only four independent

Ve will take these as $\Lambda, \Phi_{00}, \Phi_{01}$

is complex it represents two component

hese the other components of the Ricci-t

sed as:

$$= 3\Lambda + \frac{\Phi_{01}\bar{\Phi}_{01}}{\Phi}$$

$$\Phi_{00}$$

sturbed universe with the coordinate sys

$$\Lambda = \frac{\mu}{24} = \frac{A}{4\Omega^3}$$

$$\Phi_{00} = \frac{3A}{\Omega^5}$$

$$\Phi_{11} = \frac{3A}{4\Omega^3}$$

$$\Phi_{22} = \frac{3A}{4\Omega}$$

$$\Phi_{01} = \Phi_{02} = 0 \qquad ($$

values and the fact that in the undistur

the ψ^{ls} are zero, we may integrate

find the values of the spin coefficien

universe:

$$= -\frac{2}{\Omega^2} - \frac{A}{\Omega^3} + \left(\frac{A^2}{2} - \frac{A^2}{2}e^{2u}\right)\Omega$$

$$A^3\left(\tfrac{1}{2} - e^{2u}\right)\Omega^{-5} + A^4\left(\tfrac{5}{8} - \tfrac{7}{4}e^{2u} - \tfrac{1}{8}\right.$$

Ω^2

onditions

to consider radiation in a universe that
y approaches the undisturbed universe gi
Φ_{00} and Λ will then have th
lus terms of smaller order. To determi
order of Φ_{01} and Ψ_0 , there a
we may proceed. We may take the smalle
mit radiation, that is $\Psi_4 = O(r^{-1}$
terms than these in Φ_{00} , Λ
turn out to have their u derivativ
y on themselves and not on the r^{-1} coeff
, the radiation field. They are thus di
by the radiation field and will not be c

ndition that $\gamma_0^{\frown} = O(r^{-5})$. Then subs

in equations (3.10 - 27) calculate the

ed in the spin coefficients and substitut

Bianchi Identities, calculate the distu

. Further iteration does not affect the

rbances.

ese methods indicate that the boundary c

$$\Lambda = \frac{A}{4\Omega^3} + O(\Omega^{-7})$$

$$\phi_{00} = \frac{3A}{\Omega^5} + O(\Omega^{-9})$$ (

$$\phi_{01} = O(\Omega^{-7})$$ (see next section) (5

$$\gamma_0 = O(\Omega^{-7})$$ (5.

ne "uniform smoothness", that is:

$$\frac{\partial}{\partial x^j} \cdots \frac{\partial}{\partial x^i} \Lambda = O(\Omega^{-7})$$

n and Penrose, we begin by integrating t

. 10 &11)

$$= \rho^2 + \sigma\bar{\sigma} + \phi_{00}$$

$$= 2\rho\sigma + \psi_0$$

$$= \frac{3A}{4\sqrt{2}\, r^{\frac{5}{2}}} + O(r^{-3})$$

$$= \frac{\psi_0^0}{8\sqrt{2}\, r^{\frac{7}{2}}} + O(r^{-4})$$

$$= \begin{bmatrix} \rho & \sigma \\ \bar{\sigma} & \rho \end{bmatrix} \qquad \varphi = \begin{bmatrix} \phi_{00} \\ \bar{\psi}_0 \end{bmatrix}$$

$= F + O(1)$ where F is co

$= rF + O(r)$

$= O(r^{-\frac{5}{2}})$

$Y = -r\varphi F + O(r^{-\frac{3}{2}}) = O(r^{-\frac{3}{2}})$

$Y = F + O(r^{-\frac{1}{2}})$

$= rF + O(r^{\frac{1}{2}}) + E$. E is constant

$= -r^{-1}I + O(r^{-\frac{3}{2}})$

singular (The case F singular correspond
y plane or cylindrical surfaces and will
re).

$-r^{-1} + O(r^{-\frac{3}{2}}) = -2\Omega^{-2} + O(\Omega^{-3})$

$O(r^{-\frac{3}{2}}) = O(\Omega^{-3})$

$-2\Omega^{-2} + g\Omega^{-3}$

$h\Omega^{-3}$

$, = O(1)$

$$= \frac{1}{\Omega + O(1)} \left[a + \int \frac{(-A + O(\Omega^{-1}))\ (\Omega +}{\Omega + O(1)} \right.$$

$$g = -A + O\left(\frac{\log \Omega}{\Omega}\right)$$

$$(\Omega + O(1)) = -h + O(\Omega^{-1})$$

$$= O(\Omega^{-1})$$

$$= O(\log \Omega)$$

$$= O(1)$$

$$(\Omega + O(1) = O(\Omega^{-1})$$

$$h = \sigma^{0}(u, x^{i}) + O(\Omega^{-1})$$

$$(-\Omega + O(1)) = O(\Omega^{-1} \log \Omega)$$

$$g = \rho^{0}(u, x^{i}) + O(\Omega^{-1} \log \Omega)$$

n and Unti, we cannot make ρ^{0} zer

ation $r' = r - \rho^{0}$, since this wou

ndary condition $\Lambda = \frac{A'}{6\Omega^{3}} + \frac{\Lambda^{0}}{\Omega^{7}}$

e above process we derive:

$$-2\Omega^{-2} - A\Omega^{-3} + \rho^{0}\Omega^{-4} + (\tfrac{1}{2}A^{2} - 2A\rho$$

$$-5\Lambda^{4} \ldots \Omega^{-6}$$

ons of x such that:

$$B = O(x^{-2}) , \quad b = O(x^{-2})$$

matrix A is independent of x and

th positive real part. Any eigenvalue w

l part is regular. Then all solutions o

$$\frac{\partial}{\partial x} y = (A x^{-1} + B) y + b$$

s $x \to \infty$. y is a c

ons to be explained below, we will assum

at

$$\phi_{01} = O(\Omega^{-5})$$

$$\frac{\partial}{\partial \Omega} \phi_{01} = O(\Omega^{-6})$$

$$\frac{\partial}{\partial x^{j}} \cdots \phi_{01} = O(\Omega^{-5})$$

the column vector

$$
\begin{array}{ccccccccc}
 & & & & & & & & 0 \\
 & & & & & & & & 0 \\
 & & & & & & & & 0 \\
 & & & & & & & & 0 \\
 & & & & & & & & 0 \\
 & & & & & & & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -2
\end{array}
$$

b are $O(\Omega^{-1})$ expressions invol

$, \frac{\partial}{\partial \Omega}\psi_0, \frac{\partial}{\partial x^i}\psi_0, \phi_{01}$ and $\frac{\partial}{\partial \Omega}$

$= O(\Omega^{-5})$

$, \xi^4 = O(\Omega^{-2})$

$\omega = O(1)$

a null rotation of the tetrad on each nu

$$l'_\mu = l_\mu$$
$$n'_\mu = n_\mu + a\bar{m}_\mu + \bar{a}m_\mu + a\bar{a}$$
$$m'_\mu = m_\mu + a l_\mu$$

onstant alone the geodesic since the tet

ransported.

ing $a = -\frac{1}{2}\tau^0$ we may make τ'

rotation

$$\phi'_{01} = \phi_{01} + a\phi_{00}$$

e have specified the null rotation we ca

ndary condition on ϕ_{01} more severe

) . We will specify the null rotation

and in that tetrad system will impose t

dition that $\phi_{01} = O(\Omega^{-7})$ and is uniform

using this condition on ϕ_{01} and

back in equation (3.44)

$$\omega = O(\Omega^{-2} \log \Omega)$$

(3.51)

$$\psi_i = O(\Omega^{-7} \log \Omega)$$

(3.12)

$$\tau = O(\Omega^{-5} \log \Omega)$$

(3.44)

$$\omega = \omega^0 \Omega^{-2} + O(\Omega^{-3} \log \Omega$$

(3.51)

$$\psi_i = O(\Omega^{-7})$$

(3.12)

$$\tau = O(\Omega^{-5})$$

(3.44)

$$\omega = \omega^0 \Omega^{-2} + O(\Omega^{-3})$$

ating the equations used with respect to

that $\psi_i, \alpha, \beta, \tau, \xi^i, \omega$

oth.

$\left[\Omega \quad \mu \right]$

3.16; 3.17; 6.28.

$$A = \begin{bmatrix} -2 & 0 & 3A \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

b are $O(\Omega^{-2})$

$\psi_2 = O(\Omega^{-4})$

$= O(\Omega^{-2})$ (

$= O(\Omega^{-1})$

and are uniformly smo

and (3.17), we may show

$= \frac{A}{2}\Omega^{-1} + O(\Omega^{-2} \log \Omega)$

(6.28)

$\psi_2 = O(\Omega^{-5} \log \Omega)$

17)

$$-2 - A\beta^0\Omega^{-3} + \tfrac{1}{2}(3A^2\beta^0 - \rho^0\beta^0 + \bar{\beta}^0\sigma^0$$

$$\frac{A}{2}\Omega^{-1} + \frac{A^2}{4}\Omega^{-2} + O(\Omega^{-3})$$

$$^{-2} - A\left(\lambda^0 + \frac{\bar{\sigma}^0}{2}\right)\Omega^{-3} + O(\Omega^{-4})$$

$$-^2 A\Omega^3\xi^{i0} + \tfrac{1}{2}(3A^2\xi^{i0} - \rho^0\xi^{i0} - \bar{\xi}^{i0}\sigma^0)$$

$$^0 + O(\Omega^{-5})$$

$$\tfrac{1}{2}(\gamma^0 + \bar{\gamma}^0)\Omega^2 + A(1 + \gamma^0 + \bar{\gamma}^0)\Omega - \frac{3}{2}$$

$$\times(1 + \gamma^0 + \bar{\gamma}^0)\log\Omega + U^0 +$$

on $(3.55) + 2\times(\overline{3.59})$

$$+ D\phi_{21} - \delta\phi_{20} + 2\bar{\delta}\Lambda = 2\lambda\psi_1 -$$

$$\mu)\phi_{10} + 2(\beta - \bar{\alpha})\phi_{20} + 2\rho\phi_{21}$$

normality relations (2.2).

$$= X^i - (\xi^i \bar{\omega} + \bar{\xi}^i \omega)$$

$$= X^{i0} + O(\Omega^{-4})$$

$$= -(\xi^i \bar{\xi}^j + \bar{\xi}^i \xi^j) \qquad i,j =$$

$$-(\xi^{i0}\bar{\xi}^{j0} + \bar{\xi}^{i0}\xi^{j0})(\Omega^{-4} \pm 2A\Omega^{-5}) + O$$

e coordinate transformation

$$u' = u$$

$$' = r$$

$$^{i} = x^i + C^i(u, x^i)$$

$$= -X^{30}(1 + C^3_{,3}) - X^{40}C^3_{,4}$$

$$= -X^{40}(1 + C^4_{,4}) - X^{30}C^4_{,3}$$

e the coordinate freedom

$$^{i} \quad D^i(x^i)$$

adial equations of $\S 3$, relations
ion constants of the radial equations ma

tion 3.23 the term in Ω^{-1} is

$$\frac{3}{4}A(\gamma^\circ + \bar{\gamma}^\circ) - \frac{3}{4}A$$

$$\gamma^\circ + \bar{\gamma}^\circ = -1$$

(6.37)

$$U = \tfrac{1}{2}\Omega^2 + U^\circ + O(\Omega^{-1})$$

(3.50), the constant term is

$$V^\circ$$

$$V^\circ = 0$$

term in (3.47)

$$P - P_{,i} = (\bar{\gamma}^\circ - \gamma^\circ)P$$

spatial rotation of the tetrad

$$m'^{\mu} = e^{-i\phi} m^{\mu}$$

real. We take $\zeta = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{4}(x^{3^2} + x^{4^2})\right)$, the

c project factor for a 2-sphere.

term in equation (3.48)

$$^{o} = \frac{1}{2}\bar{\nabla} S e^{u}$$

$$^{o} = -\frac{1}{2}\nabla S e^{u} \qquad (7.5$$

$$= \frac{\partial}{\partial x^3} + i\frac{\partial}{\partial x^4}$$

term in (3.21)

$$\nabla\bar{\nabla} S + S\bar{\nabla}\nabla S) = -2\mu^{o} - \frac{A^2}{2}$$

$$= -\frac{A^2}{4} - \frac{S^2}{2}\nabla\bar{\nabla}\log S e^{2u}$$

$$= -\frac{A^2}{4}\left[1 + e^{2u}\right]$$

) (3.60) (3.61)

$$) \phi \qquad O(D-7)$$

$$-\frac{A^2}{8} + \frac{3U^0}{4} + \frac{\rho^0}{4} + \frac{A^2 e^{2u}}{8}$$

coordinate transformation*

$$q' = q - \int_{u0}^{u} H \, du''$$

$$H' = 0$$

$$U^0 = \frac{A^2}{6} - \frac{A^2 e^{2u}}{6} - \frac{\rho^0}{3}$$

^{4} term in (3.26)

$$-\rho^0 + 4U^0 = -\frac{A^2}{2}(1 + e^{2u})$$

$$\rho^0_{,11} - \frac{7}{3}\rho^0 + \frac{7}{6}A^2 - \frac{1}{6}A^2 e^{2u}$$

$$\rho^0 = \frac{A^2}{2}\left(1 - e^{2u}\right) + C(x^i$$

$(\bar\sigma^{\circ}_{,1} - \bar\sigma^{\circ})\, e^{u} \nabla S - \frac{1}{2} e^{u} S \nabla (\bar\sigma$

term in (3.50)

$\omega^{\circ}_{,1} = \frac{1}{2}\bar\Psi^{\circ}_{3}$

$\frac{e}{4}^{u}(S\bar\nabla\sigma^{\circ} - 2\sigma^{\circ}\bar\nabla S) + K(x^{i})e^{2u}$

term in (3.20)

${}^{\circ} + 4\omega^{\circ} - e^{u} S\bar\nabla\sigma^{\circ} = -2\sigma^{\circ}\bar\nabla S e$

$C = K = 0$

$U^{\circ} = 0$

${}^{\circ} = \frac{A^2}{2}(1 - e^{2u})$

7.16), in (6.28)

$\Psi_2 = O(\Omega^{-6}\log\Omega)$

17)

$= \frac{A}{2}\Omega^{-1} - \frac{A^2}{4}(1 + e^{2u})\Omega^{-2} + \frac{A^3}{2}($

$+ O(\Omega^{-4}\log\Omega)$

$$\phi_{00} = O(\Omega^{-1})$$

$$\Lambda = O(\Omega^{-7})$$

$$\rho_0 = O(\Omega^{-7})$$

the boundary conditions (5. 1-4)

rsurface, they will hold on succeeding h

$$\psi_4 = O(\Omega^{-2})$$

off" behaviour is therefore:

$$= O(r^{-1})$$

$$= O(r^{-2})$$

$$= O(r^{-3})$$

$\phi_{0i}^{0}\Omega^{-7} + \phi_{0i}^{i}\Omega^{-8} + O(\Omega^{-9})$

$\psi_{0}^{0}\Omega^{-7} + \psi_{0}^{i}\Omega^{-8} + O(\Omega^{-9})$

$\frac{A}{4}\Omega^{-3} + \Lambda^{0}\Omega^{-7} + O(\Omega^{-8})$

$3A\Omega^{-5} + \phi_{00}^{0}\Omega^{-9} + O(\Omega^{-10})$

$= -2\Omega^{-2} - A\Omega^{-3} + \frac{A^{2}}{2}(1-e^{2u})\Omega^{-}$

$+ [A^{4}(\frac{5}{8} - \frac{7}{8}e^{2u} - \frac{1}{8}e^{4u}) - \frac{\sigma^{0}\bar{\sigma}^{0}}{2}$

$\frac{1}{3}[A^{5}(-\frac{37}{8} + 8e^{2u} + 8e^{2u} + 2e^{4u})$

$\bar{\sigma}^{0}(3A\sigma^{0} + \psi_{0}^{0}) + \sigma^{0}(3A\bar{\sigma}^{0} +$

$O(\Omega^{-8})$

$\sigma^{0}\Omega^{-4} - (2A\sigma^{0} + \psi_{0}^{0})\Omega^{-5} + O$

$$\frac{}{4} \qquad \frac{}{4} \qquad 4\big(\frac{}{4}$$

$$\Omega^{-5})$$

$$- \frac{A}{4}^{2}(1+e^{2u})\Omega^{-2} + \frac{A}{4}^{3}(1+2$$

$$\Big(\frac{5}{8} + \frac{7}{4}e^{2u} + \frac{1}{8}e^{4u}\Big) + \frac{\sigma^{0}}{2}(\bar{\sigma}^{0}_{,1}$$

$$^{-4} + O(\Omega^{-5})$$

$$^{0}_{,1} - \bar{\sigma}^{0})\Omega^{-2} - \frac{A\bar{\sigma}^{0}_{,1}}{2}\Omega^{-3} + O(\Omega$$

$$(\bar{\sigma}^{0}_{,1} - \bar{\sigma}^{0})e^{u}\nabla S - \frac{1}{2}e^{u}S\nabla(\bar{\sigma}_{,}$$

$$\Omega^{-3})$$

$$\bar{\nabla}S\Omega^{-2} - \frac{1}{2}Ae^{u}\bar{\nabla}S\Omega^{-3}$$

$$\Big[A^{2}e^{u}\nabla S\Big(\frac{5}{2} + \frac{1}{2}e^{u}\Big) + e^{u}(\bar{\nabla}S$$

$$\frac{S}{\ }\left[\phi^{\circ}_{0i} + \bar{\phi}^{\circ}_{0i} + \psi^{\circ}_{i} + \bar{\psi}^{\circ}_{i}\right]\Omega^{-}$$

$${}^{\circ} + \frac{3}{2}\sigma^{\circ}_{,1} + -\frac{1}{2}\sigma^{\circ}_{,11}\right]\Omega^{-2} + \left[-\right.$$

$$\bar{\sigma}^{\circ}_{,11}\right]\Omega^{-3} + \left[A^{2}\left(\bar{\sigma}^{\circ} + \frac{3}{4}\bar{\sigma}^{\circ}_{,1} - \frac{3}{8}\right.\right.$$

$$\left(-\bar{\sigma}^{\circ} + \frac{3}{2}\bar{\sigma}^{\circ}_{,1} - \frac{1}{2}\bar{\sigma}^{\circ}_{,11}\right)e^{2u}$$

$${}^{\circ}_{,1} - \bar{\sigma}^{\circ})^{\frac{3}{2}}\nabla\left(S\left(\bar{\sigma}^{\circ}_{,1} - \bar{\sigma}^{\circ}\right)^{\frac{1}{2}}\right)$$

$$\bar{\psi}^{\circ}_{0}\right]\Omega^{-4} + O(\Omega^{-5})$$

$$\left(\bar{\sigma}^{\circ}_{,1} - \bar{\sigma}^{\circ}\right)\nabla S - e^{u}S\nabla\left(\bar{\sigma}^{\circ}_{,1} - \right.$$

$${}^{u}\left(-2\left(\bar{\sigma}^{\circ}_{,1} - \frac{11}{8}\bar{\sigma}^{\circ}\right)\nabla S - S\nabla\left(\right.\right.$$

$]\Omega^{\cdot} + O(\Omega^{\cdot} \log \Omega)$

$^{0}_{2} = \Big[e^{2u} \Big(\frac{S^2}{2} \bar{\nabla}\bar{\bar{\nabla}}\sigma^{0} - S(\bar{\bar{\nabla}}S)($

$+ \sigma^{0}(\bar{\bar{\nabla}}S)^{2} - \sigma^{0} S \bar{\nabla}\bar{\nabla} S \Big) -$

$^{0}_{2}$ is undetermined

$\Omega^{-7} + \psi_1^{1} \Omega^{-8} \log \Omega + \psi_1^{2} \Omega^{-8} + ($

$e^{u} \Big[S \bar{\nabla} \Big(\psi_0^{0} + \frac{15}{4} \sigma^{0} \Big) - \Big(2\psi_0^{0} +$

$- 3 \phi_{01}^{0}$

$- e^{u} (S \bar{\nabla} - 2 \bar{\nabla} S)(5A\psi_0^{0} - \psi_0^{1}$

ypersurface, it will remain zero. In t

e to continue the expansions of all quan

owers of Ω without any log terms appe

totic Group

ic has the form:

$$g^{11} = g^{13} = g^{14} = 0, \quad g^{12} = 1$$

$$g^{22} = \Omega^{2} + O(\Omega^{-2})$$

$$g^{2i} = O(\Omega^{-4})$$

$$g^{ij} = -2P^{2}\delta^{ij}\Omega^{-4} + 2AP^{2}\delta^{ij}$$

c group is the group of coordinate trans

e form of the metric and of the boundary

t can be derived most simply by consideri

$${}_{,\alpha}$$

$${}_{0} = \epsilon R_{i\alpha} K^{\alpha}_{,i} + \tfrac{1}{2}\epsilon R_{11,\alpha} K$$

$$= \tfrac{1}{2}\epsilon(R_{i\alpha} K^{\alpha}_{,3} + R_{3\alpha} K^{\alpha}_{,1} + R$$

asymptotic group we demand

$$\bar{\delta} g^{12} = \bar{\delta} g^{13} = \bar{\delta} g^{14} = 0$$

$$O(\Omega^{-2})$$

$$O(\Omega^{-4})$$

$$O(\Omega^{-6})$$

$$O(\Omega^{-7})$$

$$O(\Omega^{-9})$$

$$O(\Omega^{-7})$$

$$\therefore K^{1}_{,1} = 0$$

$$K^{1} = K^{01}(x^{i})$$

$$K^{3}_{,2} + g^{33} K^{1}_{,3} + g^{34} K^{1}_{,4}$$

$$K^{3} = K^{03}(u, x^{i}) - 4r^{-1}P^{2} K^{i}_{,3} +$$

$$(\Omega^{-4}) = K^{3}_{,1} + 2r K^{3}_{;1} + O(\Omega$$

$$\therefore K^{03} = K^{03}(x^{i})$$

$$^{i2} K^{j}_{,2} + g^{j2} K^{i}_{,2} + g^{ik} K^{j}_{,k} + g$$

$$g^{ij}_{;k} K^{k}$$

$$^{03}_{,4} = -K^{04}_{,3}$$

$$_{3} = K^{04}_{,4}$$

$$S_{,3} K^{03} - S_{,4} K^{04} - 2S K^{1} = 0$$

3) imply that K^{0i} is an analytic functi

This is a consequence if the fact tha

$$x^4 = \frac{a(x^3 + ix^4) + b}{c(x^3 + ix^4) + d}$$

$$ad - bc = 1$$

parameters a, b, c, d are given K^1 is u
(8.14) K^2 is also uniquely determine
c group is isomorphic to the conformal g
s. Sachs[7] has shown that this is iso
neous Lorentz group. It is also however
of motions of a 3-space of constant nega
ch is the group of the unperturbed Rober

Thus the asymptotic group is the same
undisturbed space. It is not enlarged b
adiation. This is interesting because i
tational radiation in empty, asymptotica
ns out that the asymptotic group contain
imensional inhomogeneous Lorentz group,

bserver would measure

city vector $\underset{m}{V}$ of an observer moving w

$= \Omega^{-1} + O(\Omega^{-5})$

$= \frac{1}{2}\Omega + O(\Omega^{-3})$

$= O(\Omega^{-3})$

$= O(\Omega^{-3})$

(

projection of the wave vector l^{a} in

st-space (the apparent direction of the

$$= \overset{2}{\underset{m}{\delta}} - \underset{m}{V}\,\underset{1}{V}$$

$$= \Omega^{-2} + O(\Omega^{-5})$$

's orthonormal tetrad may be completed by

nit vectors $\underset{m}{s}$ and $\underset{m}{t}$

$$(\Omega^{-4}) \qquad \underset{1}{t} = O(\Omega$$

$$\Omega^{-2}) \qquad \underset{2}{t} = O(\Omega$$

$$\underset{3}{t} = \frac{-i}{\sqrt{2}}$$

$$\underset{4}{t} = \frac{i}{\sqrt{2}}$$

$$= (V^{\alpha}, q^{a}, s^{a}, t^{\alpha})$$

the relative accelerations of neighbour

ne observer may determine the 'electric'

$$\begin{matrix} 0 & -1 \end{matrix}\Bigg] \qquad \Bigg[\begin{matrix} 0 & 1 & 0 \end{matrix}\Bigg]$$

$$\begin{matrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{matrix}\Bigg] O(\Omega^{-5}) + \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} O(\Omega$$

$$\begin{matrix} 0 & 0 \\ & 0 \\ & -\tfrac{1}{2} \end{matrix}\Bigg] O(\Omega^{-6})$$

e compared to the behaviour for asymptot

r which

$$\begin{matrix} 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{matrix}\Bigg] O(r^{-1}) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} O(r^{-1})$$

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Einstein equations without cosmological c
d, a Robertson-Walker model can 'bounce'
y only if the pressure is less than minus
nsity. This is clearly not a property po
tter though it might be possessed by a fi
rgy density like the 'C' field. However
tum-mechanical difficulty associated with
negative energy density, for there would
revent the creation, in a given volume of
infinite number of quanta of the negative
corresponding infinity of particles of po
we therefore exclude such fields, all Rob
s must be of the 'big-bang' type, that i
larity in the past and maybe one in the
t has been suggested [1] that the occurrenc
arities is a consequence of the high deg
the Robertson-Walker models which restri

mental Equation

ansion $\theta = V^a{}_{;a}$ of a time-like

with unit tangent vector V^a obeys equ

:

$$_a V^a = -\tfrac{1}{3}\theta^2 - 2\sigma^2 + 2\omega^2 - R_{ab} V$$

will be said to be a singular point or

time-like geodesic congruence if θ for th

on γ at q . A point q will be said

o a point p along a geodesic γ if i

nt on γ of the congruence of all time-l

rough p . A point q will be said t

ike hypersurface H^3 if it is a sin

uence of geodesic normals to H^3 . An a

of conjugate points may be given as fol

Ds

$$\frac{D^2 K^a}{Ds^2} = R^a{}_{bcd} V^b V^c K^d$$

n orthonormal tetrad $\underset{m}{e}{}^a$ parallelly tra

$\underset{4}{e}{}^a = V^a$ we have

$$= -\underset{m}{\overset{n}{a}} \underset{n}{K}$$

$$= \overset{n}{e}{}^a \underset{m}{e}{}^b R_{acbd} V^c V^d$$

(4) will be called a Jacobi field. Th

independent solutions. Since V^a and

e other six independent solutions of (4)

nal to V^a. Then q is conjugate to p

if, and only if, there is a Jacobi fiel

s at p and q. This may be shown as f

elds which vanish at p may be regarded written

$$= A_{mn}(s) \left. \frac{dK^n}{ds} \right|_p$$

$$\frac{A_{mn}}{s} = V_{m:r} A^r_n$$

(s) will be positive definite. There

ld vanishing at p and q if, and only

$$= \exp\left(\int_p^s V_{m:n}\, ds'\right)$$

$$= \frac{1}{\det(A_{mn})} \frac{d}{ds} \left(\det(A_{mn})\right)$$

$$\frac{A_{mn}}{{}^2} = -a_{mr} A^r_n$$

$\frac{d}{ds}(A_{mn})$ is finite

infinite where and only where $\det(A_{mn}$

ingular point of a null geodesic congruen
is infinite.

ition that the pressure is greater than
density may be stated more generally as
.

$E > 0, \quad E > \frac{1}{2} T,$ for an
4-velocity ω^a, where $E = T_{ab}\, \omega^a \omega$
density in the rest-frame of the observ
is the rest-mass density.

n (a) will be satisfied by a perfect flu
and pressure $p > -\frac{1}{3}\mu$. It implies $R_{ab} V^a$
or null vector V^a. Therefore by eq
ny time-like or null irrotational geodes
st have a singular point on each geodesi
ne distance. Obviously if the flow-line
al geodesic congruence, there will be a
t the singular points of the congruence

considering models that are spatially ho
pic (that is, they have a three paramete
ransitive on a space-like hypersurface)
low may have rotation, acceleration and
ould seem to be the possibility of non-s
Shepley[4] has investigated one particular
model containing rotating dust and has s
ays a singularity. Here a general resul

ust be a singularity in every model whic
) and,
ists a G_r of motions on the space or o
vering space *, $r \geq 3$ which is transit

ion 5

ace-like surface but space-time is not s
gy-momentum tensor is that of a perfect

$$\ldots = \frac{e(R_{;a})\,R_{;a}}{\sqrt{f}}$$

$$\ldots^{;a} = f > 0$$

is an indicator = +1 if $R_{;a}$ is past di
= -1 if $R_{;a}$ is future

congruence of geodesic irrotational time
condition (a), $R_{ab}V^aV^b > 0$.
congruence must have a singular point
equation 1) either in the future or in
the homogeneity, the distance along each
the singular point must be the same for ea
surfaces of transitivity remain space-lik
ate into, at the most, a 2-surface C^2 wh
defined . Let M be the subset of the f
which intersect C^2 . Let ℓ be th

do not remain space-like, there must be
rface which is null - call this S^3. At S
if $R_{;a}$ is zero, we can take any other sca
n the curvature tensor and its covariant
all be zero if space-time is not station
geodesic irrotational null congruence on
or l^a where l_a $R_{;a}$. Then by equation
e a singular point of each null geodesic
e affine distance either in the future or
surface of these singular points will be
e same argument used before shows that t
nite and there is a physical singularity
urface of homogeneity, the whole of S^3 w
it is not meaningful to call it null or
this case from the case where the surfac
remain space-like.

ditions (a), (b), (c) may be weakened in neous space section.

ist equations of state such that the Cau

f H^3 is determinate.

ceeding space-like surfaces of constant

us and much the same proof can be given

ngular models satisfying (a), (b), (c),

property of perfect fluids that has bee

proof is that they have well defined flo

of which implies a physical singularity.

property will be possessed by a much mo

ds. For these, we define the flow vecto

envector (assumed unique) of the energy-

we can replace condition (c) on the nat

the much weaker condition (e).

odel is singularity-free, the flow-lines

-like congruence with no singular points

h each point of space-time.

n (e) will be satisfied automatically if n cannot prevent the singularity.

f interest to examine the nature of the

eneous anisotropic models since this is

representative of the general case than

c models. It seems that in general the

ne direction,[5] that is, the universe wi

surface. Near the singularity, the volum

to the time from the singularity irrespe

nature of the matter. It also appears

e particle horizon is different. There

orizon in every direction except that in

is taking place.

<u>ties in Inhomogeneous Models</u>

t and Khalatnikov[6] claim to have proved

tion of the field equations will not hav

Their method is to contract a solution

ne would expect collapse in one direction
In fact their claim has been proved false
the case of a collapsing star using the
trapped surface'. A similar method will
e the occurrence of singularities in 'ope
els.

d 'Closed' Models

hod used by Penrose to prove the occurre
l singularity depends on the existence o
Cauchy surface. A Cauchy surface will be
mplete, connected space-like surface that
very time-like and null line once and onc
es possess a Cauchy surface: examples o
include the plane-wave metrics,[8] the God
pace.[10] However none of these have any
. Indeed it would seem reasonable to de

o: there exist possible topologies for
act. However, the following statements
e topology of the surfaces t = constant.
urvature is negative, K = -1, the univer
e is non-compact and is diffeomorphic to
ology can be obtained by identification
any other topology will not be simply c
ct, must have elements of infinite order
roup. Further if compact, they can have
ons.[12]

urvature is zero, K = 0, the universal c
There are eighteen possible topologies.
have a G_3 of motions and Betti numbers,

urvature is positive, K = +1, the univer
e is S^3. Thus all topologies are compac
are all zero.[12]

have negative or zero curvature.

Trapped Surface

e a 3-ball of coordinate radius r in a 3

.) in a Robertson-Walker metric with K =

outward directed unit normal to T^2, the

and let V^a be the past directed unit nor

the outgoing family of null geodesics w

orthogonally. At T^2, ρ , their converge

$$\tfrac{1}{2}\left(V_{a;b} + q_{a;b}\right)\left(s^a s^b + t^a t^b\right)$$

are unit space-like vectors in H^3 ortho

each other,

$$= \frac{2}{R}\left[\sqrt{\frac{\mu}{3} - K} \;-\; \frac{1}{r}\sqrt{1 - Kr^2}\right]$$

1 K = 0 or -1, by taking r large enough,

ve at T^2. Therefore, in the language of

1 trapped surface.

It reaches a maximum proper radius (ρ
erges again to the singularity ($\rho > 0$
of the converging light cone and the sur
d trapped surface T^2. If the red-shift
3C9 is cosmological then it will be bey
if we are living in a Robertson-Walker
normal matter. However, the assumption
nd isotropy in the large seem to hold ou
C9. Thus there is good reason to believ
does in fact contain a closed trapped su
pointed out that the possession of a clo
ce is a large scale property that does n
local metric. Thus a model that had loca
tion and shear but was similar on a larg
t time to a Robertson-Walker model would
d surface.
g Penrose it will be shown that space-ti

to the past of H^3 that can be joined by
directed time-like line to T^2 or its in
be the boundary of F^4. Local consideratio
is null where it is non-singular and is
the outgoing family of past directed nul
ch have future end-point on T^2 and past e
ore a singular point of the null geodesic
Since at T^2, the convergence, $\rho > 0$ a
$L^b \geq 0$ by (f), the convergence mu
te within finite affine distance. Thus
ct being generated by a compact family o
nce B^3 will be compact. Penrose's metho
ows: approximate B^3 arbitrarily closel
-like surface and project B^3 onto H^3 by t
is surface. This gives a many-one conti
^{3} into H^3. Since B^3 is compact, its imag
act. Let $d(Q)$ be the number of points o

ce-like surface is possible if we adopt

nature of the matter, then B^3 may be pro

y one-to-one onto H^3 by the flow-lines.

to a contradiction since B^3 is compact

orizons

~~above proof it was necessary to demand~~

~~chy surface otherwise the whole of B^3 mi~~

~~rojected onto H^3. We will define a semi~~

~~G.s) as a complete connected space-like~~

~~ects every time-like and null line at mo~~

~~will be a Cauchy surface for points near~~

~~will intersect every time-like and null~~

~~se points. However, further away there m~~

~~which it is not a Cauchy surface. Let~~

~~ts for which H^3 is a Cauchy surface and~~

~~y of these points. Q^3, if it exists, w~~

~~ull cone of the point in Minkowski space~~

~~rms the Cauchy horizon~~.

~~itions (e) and (g) hold, then a model wi~~

~~s, H^3 must have the topology: $H^3 \times E^1$~~.

~~there were a region V^4 through which th~~

~~s intersecting H^3, then V^3 the boundary~~

~~ne-like surface generated by flow-lines~~

~~. Proceeding slong these flow-lines in~~

~~their intersection with H^3, we must rea~~

~~the generator since V^3 does not inters~~

~~sence of this end-point contradicts (e)~~

~~ngularity of the flow-line congruence.~~

~~and every point has a flow-line through~~

~~H^3. Thus we have a homeomorphism of th~~

~~by assigning to every point of the space~~

~~s the flow-line from H^3 and the point o~~

~~ect $\mathcal{Q}^3$. Thus $\mathcal{Q}^3$ is homeomorphic to H~~

~~f condition (f) holds every null geodesi~~

~~t least one end-point. This must be in~~

~~way from H^3 since in the direction towar~~

~~tor must be unbounded. This however is~~

~~compact. Thus H^3 is a Cauchy surface~~.

ities in 'Closed' Universes

is a singularity in every model which s
d (i).

e exists a compact Cauchy surface H^3 who
has positive expansion everywhere on H^3

e proof it is necessary to establish a c
sume that space-time is singularity-free
esult is quoted without proof, it may re
m lemmas proved in reference 11.

pq, any irrotational geodesic congruenc
c γ must have a singular point on γ i
s a point on M^3 and γ is the geodesic n
q, then a point conjugate to q along γ
a point conjugate to M^3.

is a complete connected space-like surfa
every time-like and null line from a poi
a function over M^3 as the square of the
om p which is taken as positive if the g
nd negative if the geodesic is space-lik
rld function σ with respect to p. For
es $\sigma \geqslant 0$, σ will be a continuou
ti-valued) function over M^3. A time-lik
p will be said to be critical if it co
of σ for which $\sigma_{;\mu} \xi^{\mu}_{i} = 0 \, (i = i,$
u are three independent vectors in M^3.
geodesic must be orthogonal to M^3. A g

x conjugate to M^3 but no point conjugate to q

is the intersection of γ and M^3.

f_m and g_m be the Jacobi fields along γ which

respectively. They may be written

$$f_m = A_{mn}(s) f^n/q ,$$

$$g_m = B_{mn}(s) g^n/q .$$

$$h^m \left(\frac{dA_{mn}}{ds}\Big|_q - \frac{dB_{mn}}{ds}\Big|_q \right) h^n ,$$ must be posi

since if it were negative for any h^n by taking a
mb

, it would be possible to have a point y on γ b

e to x before a point conjugate to p . If i

ld be conjugate to p . This shows that the su

ant geodesic distance from p lies nearer to p i

n than the surface ~~of~~ q of constant geodesic dis

oes. Since x is conjugate to M^3 the surface a

geodesic distance from p lies closer to ~~M~~ P_3 in

3

~~st be some direction $\overset{m}{K}$ for which~~

~~$\overset{n}{K} \frac{d}{ds} A_{mn} < \overset{m}{K} \overset{n}{K} \frac{d}{ds} B_{mn}$~~

~~$\overset{n}{K} V_{m;n} < \overset{m}{K} \overset{n}{K} w_{m;n}$~~

~~w is the unit tangent vector of the con~~

~~through p. Thus in the direction K the~~

~~eodesic distance from p lies closer to p~~

~~3 does. Therefore γ is not maximal~~.

compact or if the intersection of all t

s with M^3 is compact, σ must have a ma

here must be a geodesic normal to M^3 thr

γ . We use this to prove another len

s to the future (past) on a time-like ge

q, beyond a point z conjugate to q,and

act Cauchy surface H^3 through q, then th

er time-like geodesic from p to q longer

) directed time-like and null line from

not in F^4, they must also intersect J^3.

section of J^3 and these lines. Since H^3

must be compact. Consider the function

over K^3. Its maximum must lie in the c

But, by the previous lemma γ is not max

cal considerations show that a singular

J^3 cannot be a maximum of σ . Thus t

e of σ must occur for a geodesic from p

This must also be a goedesic from p to

er than γ .

hese two lemmas the theorem may be prove

past) directed normals to H^3 are converg

n H^3, there must be a point conjugate to

nce along each future (past) directed ge

β be the maximum of these distances.

n a future (past) directed geodesic norm

e maximal by the first lemma. If however
onjugate to q along λ in qp, then there m
lesic from q to p by the second lemma. T
odesic of maximum length from H^3 to p.
on which shows that the original assumpt
non-singular must be false.
of could also be used to show the occurr
ity in a model with a non-compact Cauchy
the expansion of its normals was bounde
provided that the intersection of the C
all the time-like and null lines from a

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